Controlling global stochasticity in the standard map

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A method for controlling global stochasticity in two-dimensional Hamiltonian systems is proposed in a model of the standard map. We demonstrate that this control method can stabilize global stochasticity into regular motion running in limited regions. The method is robust under the presence of weak external noise. Noise-induced intermittency can be found in the case of large noise strength. This is a new type of intermittency similar to the recently discovered in-out intermittency.

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Problems in many branches of physics can be reduced to the study of two-dimensional measure-preserving (Hamiltonian) mappings. Maps are intriguing, while the corresponding equations are simple and deterministic, their solutions are either ordered or chaotic [1]. An example of these maps, the standard map, corresponding to the kicked rotor, is one of the most widely studied, and it is given by

$$p_{n+1}=p_n-\frac{K}{2\pi}\sin(2\pi x_n) \mod 1 \tag{1}$$

$$x_{n+1}=x_n+p_{n+1} \mod 1.$$

The standard map has become a paradigm for the study of properties of chaotic dynamics in Hamiltonian systems. For $K=0$ the map is integrable and degenerate. An orbit with a rational winding number $\omega(p_0)=N/M$ [2] will be periodic with period M. For K less than the threshold value $K_c \cong 0.9716, \ldots,$ [1], which corresponds to winding number $\zeta = (\sqrt{5}-1)/2$ (known as golden mean), the phase space is divided into regions separated by Kolmogorov, Arnol'd, and Moser (KAM) [3], which encircle the p,x torus horizontally, so that motion in p is bounded to limited regions. As K increases the KAM surfaces become sparse, they are completely destroyed for $K>K_c$. At this point the motion in p is unbounded and global stochasticity, or global chaos, sets in.

The winding number of the last KAM surface $\zeta=(\sqrt{5}-1)/2$ [2] can be represented by the infinite continued fraction

$$\omega=0+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}$$

or $\omega=[0,1,1,\ldots]$ [2]. Therefore the last KAM tori will be approximated by periodic orbits of winding numbers given by the approximation of the golden mean. The rational approximants, i.e., the corresponding winding numbers of these periodic orbits, are $\omega_0=\frac{0}{1}$, $\omega_1=\frac{1}{1}$, $\omega_2=\frac{1}{2}$, $\omega_3=\frac{2}{3}$, $\omega_4=\frac{3}{5}$, $\omega_5=\frac{5}{8}$, etc. These orbits will remain stable when K decreases to zero adiabatically. On the other hand, as we increase K from K_c, fewer and fewer orbits will remain stable. The last one, with winding number ω_0, becomes unstable at $K=4$. The existence of island chains corresponding to these periodic orbits play an important role in the global stochasticity in the range of $K_c<K<4$. Not much attention has been given to the control of global stochasticity in this range of K, which is very important for practical applications. For instance, if we refer to the tokamak problem, this kind of chaos controlling will correspond to the confinement of magnetic field lines to the toroidal chamber, instead of diffusing globally [4].

In this work we propose a method to control global stochasticity by means of pinning the variable p to that of the nearest rational approximants, when $K=K_c$. The procedure is as follows: under the application of control the standard map becomes

$$p_{n+1}=p_n-\frac{K}{2\pi}\sin(2\pi x_n)+(p_n^*-p_n) \mod 1, \tag{2}$$

$$x_{n+1}=x_n+p_{n+1} \mod 1,$$

where p_n^* is the pinning value. In order to control the chaotic motion onto a periodic orbit, we consider the first 15 approximants to the golden mean. Extended simulations show that the first six of them are enough to take the influence of all the 15 solutions in the controlling task. The value of p^*, using the first six approximants plus the golden mean, are $\{\frac{0}{1}, \frac{1}{3}, \frac{3}{8}, (\sqrt{5}-1)/2, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, (\sqrt{5}+1)/2, \frac{5}{8}, \frac{2}{3}, \frac{1}{1}\}$. The other four arise from the symmetries in p around $p=\frac{1}{2}$. At the nth step, we calculate $\min(|p^*-p_n|)$ from all the 15 approximants, this value of p^* is inserted into Eq. (2) as p_n^*, then we iterate one step to obtain (p_{n+1},x_{n+1}). With this new value of p_{n+1} we repeat the process until we reach a stable value.

The pinning method described by Eq. (2) changes the dynamics of the original system while it controls its global chaotic motion into a regular one. First, under the pinning control, the conservative system changes into a dissipative one. Second, the maximum change induced by the control can be up to 16% of the motion in p, so this is not a weak perturbation to the original system. Third, the pinning is not

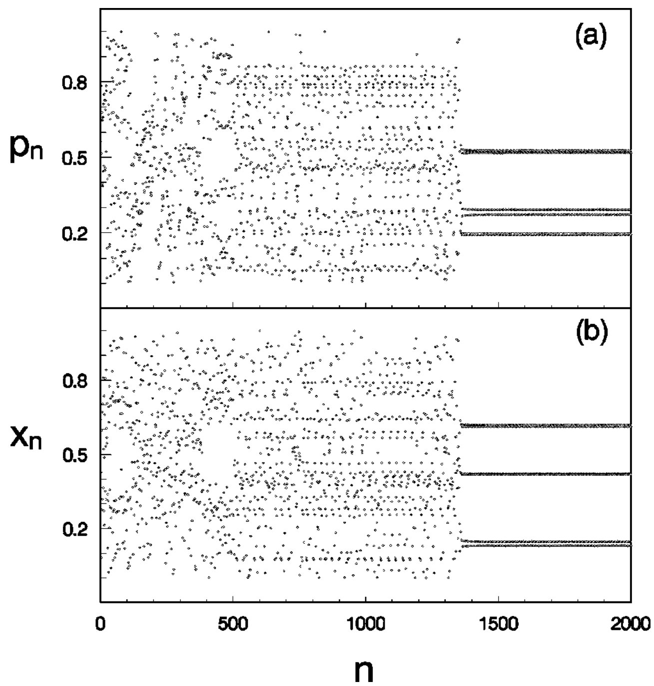

FIG. 1. The stabilized global chaotic motion. (a) p_n vs n; (b) x_n vs n. $K=1.8$, and from $n=500$, the control is switched on.

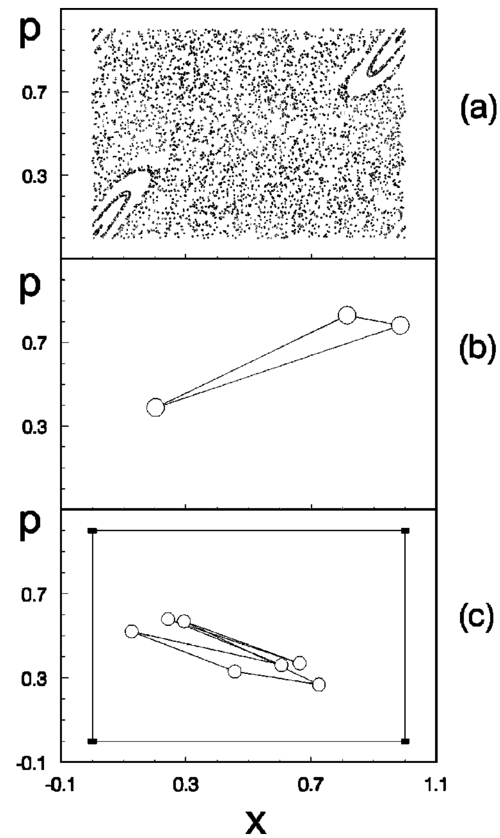

FIG. 2. The typical orbits in the phase space (x,p). (a) The system without control, (b) and (c) the system under control. (a) and (b) $K=3.9$, (c) $K=1.5$, and 10^7 iterations have been cut. (c) There are two solutions, one is indicated by circles, the other is indicated by squares.

onto any certain periodic orbit, but onto p^* only (no certain x^* is pinned). In other words, this is not a pinning treatment in the exact meaning. As a consequence, the regular motion that results from the pinning control being applied generally does not correspond to the previously unstable periodic orbits, and this is confirmed in simulations.

Figure 1 shows a global chaotic motion being controlled into regular period-six orbit. In Fig. 2 we show how the phase space is changed by turning on the control by pinning. All the chaotic orbits diffusing globally, as well as the island chains, turn into localized regular orbits. Figure 2(a) depicts the phase space of the standard map with $K=3.9$. Figure 2(b) shows the effect of control: all orbits, regular and chaotic turn, into a period-three orbit. For different values of K, the phase space structure under control is different, in some cases, as the one shown in Fig. 2(b), the solution is unique, while in other cases there may be coexisting periodic solutions as shown in Fig. 2(c). The time needed to control the system, relaxation time, varies as much as seven orders of magnitude depending on the initial conditions and the value of K. For the cases of Fig. 2, the relaxation time is of the order of 10^7 time steps.

We believe that the existence of island chains of stability play an important role in the success of controlling chaotic orbits. To verify this assertion we apply the pinning method of control to a system with $K>4$. In our calculation we noticed that our method always controls the chaotic orbits when $K<4$, and island chains of stability exist. On the other hand, for $K>4$ the success of control depends on the initial conditions and different value of K. For some initial conditions, it is successful as in Fig. 3(a); while for many other initial conditions or the same initial conditions but different values of K, it is invalid as shown in Figs. 3(b) and 3(c). The motions in Fig. 3 have been checked in 10^9 iterations, and we can assure they reflect the behavior of the system under control. We know that $K=4$ is the bifurcation point for the last island losing stability, while the approximants in the pinning set correspond to the island chains one by one. Since the method is always valid for $K<4$ while for $K>4$ it is not successful in all cases. The complete success of pinning control depends on whether the pinning set has elements corresponding to the island chains that remain stable.

The preceding simulation shows that the pinning method is able to control global stochasticity into regular and local motion. It remains to show if the method is robust under the presence of noise, which will be important for applications. To find this out, we consider the effect of added noise; thus

$$p_{n+1}=p_n-\frac{K}{2\pi}\sin(2\pi x_n)+(p_n^*-p_n)+\rho\xi_n \mod 1, \tag{3}$$

$$x_{n+1}=x_n+p_{n+1}+\rho\eta_n \mod 1,$$

$$\langle\xi_n\xi_{n'}\rangle=\langle\eta_n\eta_{n'}\rangle=\delta(n-n'),\quad \langle\xi_n\eta_{n'}\rangle=0,$$

$$\langle\xi_n\rangle=\langle\eta_n\rangle=0,$$

where ξ_n and η_n are Gaussian white noise generated by using the Box-Müller method [5], and ρ denotes the intensity of external noise.

The results are shown in Fig. 4, where we have considered the cases of Fig. 1, with $K=1.8$, under the presence of

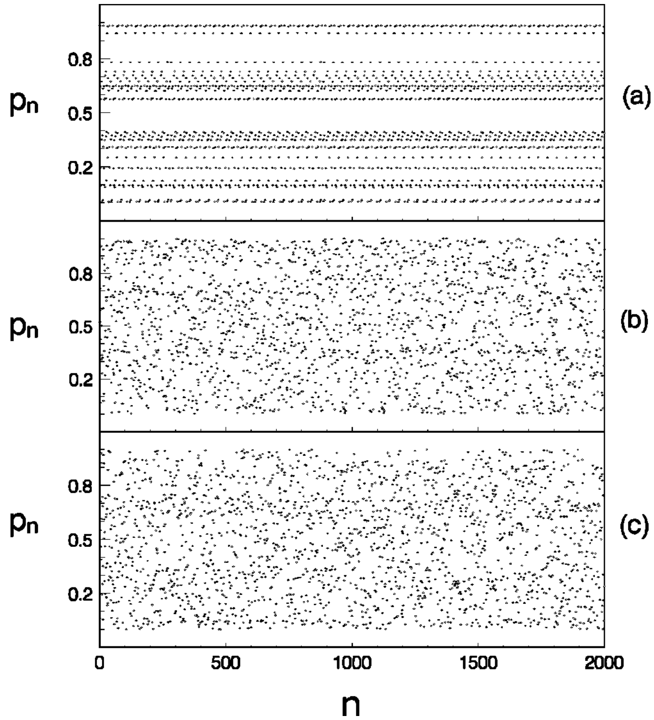

FIG. 3. The typical motions of the system under control when $K>4$ and after the relaxations have been cut off. (a) and (b) Different initial values with $K=4.1$. (c) The same initial condition as that in (a) while $K=4.5$.

noise. The intensity of the noise in Figs. 4(a) and 4(b) is 5.0×10^{-4} and 2.25×10^{-3}, respectively. Here we plot only the variable p. Comparing Fig. 4(a) with Fig. 1(a) we observe that the final motion is rather well resolved onto a noisy period-six orbit. But most important, the motion is

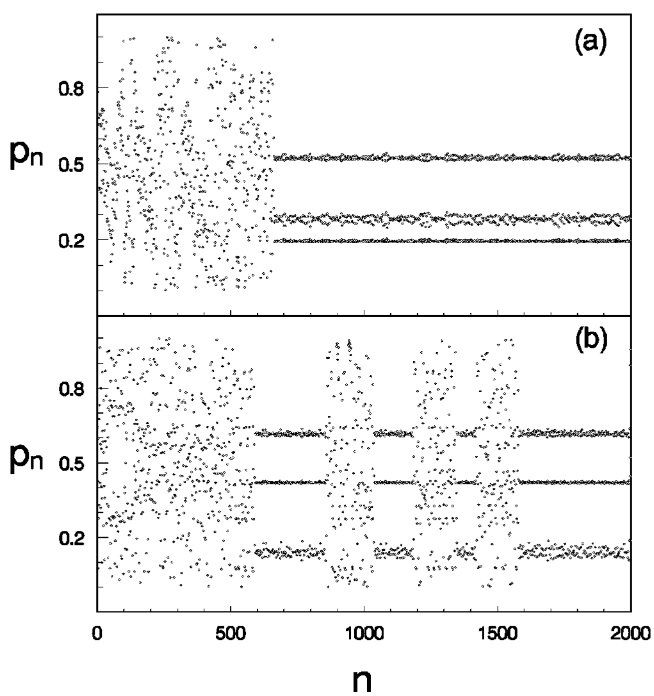

FIG. 4. The effect of noise on the controlled system corresponding to Fig. 1(a). The noise intensity is 5.0×10^{-4} and (b) it is 2.25×10^{-3}.

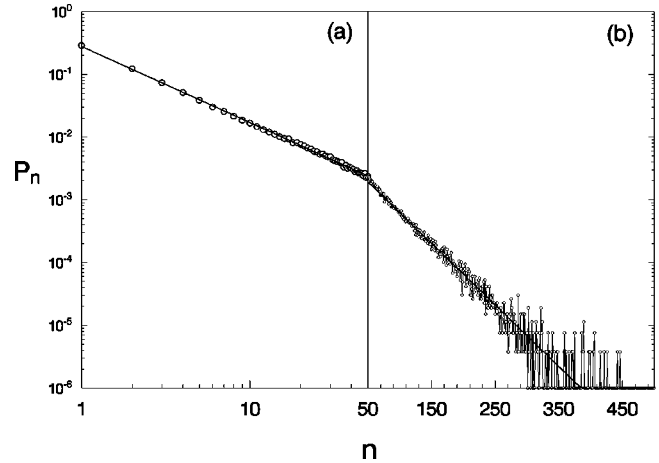

FIG. 5. The distributions of noisy periodicity phase (NPP) with different length n corresponding to the evolution in Fig. 4(b), where the plot is $\ln P$ vs $\ln(n)$ for $n<50$, and $\ln P$ vs n for $n>50$, to evidence the scaling laws.

localized in phase space, it remains within a small neighborhood of the noise-free orbit and it does not wander over the whole phase space. Therefore we conclude that the pinning method is robust against weak external noise.

To estimate the robustness, we increase the strength of the noise, and show the result in Fig. 4(b). Noise in this case, $\rho=2.25\times10^{-3}$, produces intermittent incursions into global stochasticity. In order to better characterize the noise-induced intermittent behavior we calculate the probability of the laminar phase, i.e., noisy periodic phase (NPP) with length n in the ensemble of NPPs with different length, by the definition: $P_n=M_n/N_{total}$, where N_{total} is the total number of segments of the laminar NNP, and M_n is the number of those with length n. Figure 5 shows how P_n scales with n. For small segments of the laminar NPP, $n<50$, it obeys a power law scaling, while for large segments the distribution is roughly exponential. The error increases with n due to computation limitations. This type of intermittency has similar characteristics to the recently discovered in-out intermittency [6–8], which also shows a phase transition, but the power law decays with an exponent $-\frac{3}{2}$, while in our case the exponent is approximately $-\frac{5}{4}$, which can be seen by comparison of our data with the line $P_n=-\frac{5}{4}n+\text{const}$. The correlation coefficient for the linear fitting in Fig. 5(a) is -0.9994, and the standard error of the slope (Sb) is 0.0063. In-out intermittency is a generalization of on-off intermittency and it is expected to be generic for axisymmetric dynamo settings [6]. The possible mechanism for in-out intermittency, as well as that for the intermittency in our case, is not known and it will be the subject of future research.

In conclusion, we propose a method to control global stochasticity by pinning a variable to successive approximants of periodic orbits. The method is described by applying it to the standard map, where the results show that it is effective in the control task. Global stochasticity (global chaos) can be controlled into localized regular motion and this can be achieved for a moderately large stochastic parameter range, $K\in(K_c,4)$.

As is well known, chaos in Hamiltonian systems has a different character from that in dissipative systems. All Hamiltonian systems, especially a two-dimensional system, have the common feature: when all KAM surfaces have been destroyed and stochasticity is unbounded, the island chains of stability will still play an important role in the complicated dynamics of the systems. We have shown numerically that when the island chains of stability exist our method is successful in controlling stochasticity. On the other hand for $K>4$, when the islands had all disappeared the success of our task depends on the initial condition. So, if the pinning set has elements corresponding to which the island chains remain stable, the pinning method is always effective.

We also show that the method is robust under the presence of weak external noise. Noise-induced intermittency is found in the controlled system when the strength of the external noise is large. The distribution of laminar phases exhibits a roughly $-\frac{5}{4}$ power law decay for small length and an approximately exponential tail for large length of the laminar phases.

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